Dependence of the vacuum energy on spherically symmetric background fields

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The vacuum energy of a scalar field in a spherically symmetric background field is considered. Based on previous work [Phys. Rev. D **53**, 5753 (1996)], the numerical procedure is refined further and applied to several examples. We provide numerical evidence that repulsive potentials lead to positive contributions to the vacuum energy. Furthermore, the crucial role played by bound states is clearly seen.

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I. INTRODUCTION

The calculation of vacuum energies in the context of quantum field theory under external conditions plays an important role in several areas of modern theoretical physics. External conditions may be generated by classical fields (for example, gravitational [1] and electromagnetic fields, monopoles [2,3], sphalerons [4], or electroweak Skyrmions [5]), or they may be realized through boundaries at which the quantum field has to satisfy certain boundary conditions (see, for example, Ref. [6]). External fields most naturally appear as classical solutions of some physical nonlinear equations, but they may also be viewed as a localized property of space representing a distribution of matter within which the quantum field exists. For external fields realized as boundaries this situation is usually called the Casimir effect. There are a number of generalizations towards weakened boundaries, e.g., by introducing semihard walls [7] or transparent boundaries realized by delta functions [8]. For those interested in such calculations see, e.g., Refs. [9,10].

A formalism to deal with arbitrary background fields has been developed in Ref. [11] for a background depending on one cartesian coordinate and in Ref. [12] for a general spherically symmetric background. However, these methods had been applied to a square well potential only where the interesting quantities can be expressed in terms of Bessel functions.

The aim of the present work is to show that this formalism is well suited for arbitrarily shaped spherically symmetric background fields. We investigate in detail different types of background fields as examples. It is observed that the main feature of the vacuum energy is determined by the number (and depth) of bound states. As long as these are kept fixed when varying parameters the general behavior of the energy is the same within the class of examples considered. At the same time we find that the presence of bound states does not fix the sign of the ground state energy. Similar calculations, however, with a less extent of details and

using other techniques, have been performed earlier, e.g., in Refs. [13] and [14].

This paper is organized as follows. In Sec. II the basic notations of the method used are explained. In Sec. III the ''asymptotic'' contribution is calculated. In Sec. IV the numerical procedure is developed in detail followed by a section containing the examples calculated. The results are discussed in the conclusions.

II. THE MODEL, ITS RENORMALIZATION AND SOME BASIC RESULTS

We consider the theory described by the Lagrangian

$$L = \frac{1}{2}\Phi(\Box - M^2 - \lambda\Phi^2)\Phi + \frac{1}{2}\varphi(\Box - m^2 - \lambda'\Phi^2)\varphi. \tag{1}$$

Here the field Φ is a classical background field. By means of

$$V(x) = \lambda'\Phi^2 \tag{2}$$

it defines the potential in Eq. (1) for the field $\varphi(x)$, which should be quantized in the background of $V(x)$.

Our interest is in the vacuum energy of the theory which is defined as half the sum of the one particle energies of the field $\varphi(x)$,

$$E_\varphi[\Phi] = \frac{1}{2}\sum_{(n)}(\lambda_{(n)}^2 + m^2)^{1/2-s}\mu^{2s}, \tag{3}$$

which is the ground state energy of the quantized field Φ according to the well known formula $E_0 = \Sigma_{(n)}(\hbar\omega_{(n)}/2)$. Here s is the regularization parameter with $s \to 0$ at the end and μ is an arbitrary mass parameter. The $\lambda_{(n)}^2$ are the one-particle energy eigenvalues determined through

$$[-\Delta + V(x)]\phi_{(n)}(x) = \lambda_{(n)}^2\phi_{(n)}(x). \tag{4}$$

For the moment a finite volume is assumed in order that the Hamiltonian has discrete eigenvalues.

The spectral function appearing in Eq. (3) is related to the zeta function of the wave operator (4)

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$$\zeta_V(s) = \sum_{(n)} (\lambda_{(n)}^2 + m^2)^{-s} \tag{5}$$

by

$$E_\varphi[\Phi] = \frac{1}{2} \zeta_V(s-1/2) \mu^{2s}. \tag{6}$$

The ultraviolet divergences contained in the ground state energy (3) need to be renormalized. The divergences are known and best expressed in terms of the heat kernel coefficients A_i related to the operator defined in Eq. (4). Using these the divergent part of the ground state energy reads

$$E_\varphi^{\mathrm{div}}[\Phi] = -\frac{m^4}{64\pi^2} \left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - \frac{1}{2} \right) A_0$$

$$+ \frac{m^2}{32\pi^2} \left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 1 \right) A_1$$

$$- \frac{1}{32\pi^2} \left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 2 \right) A_2, \tag{7}$$

with the coefficients $A_1 = -\int d\vec{x}\, V(x)$, $A_2 = \frac{1}{2}\int d\vec{x}\, V(x)^2$, and A_0 is the spatial volume. The contribution of A_0 will be dropped from now on without further comment because it does not depend on the background.

The renormalization procedure is determined by the definition

$$E_\varphi^{\mathrm{ren}} = E_\varphi[\Phi] - E_\varphi^{\mathrm{div}}[\Phi] \tag{8}$$

and the corresponding counterterms can be absorbed into a redefinition of the parameters M^2 and λ of the classical background field with energy

$$E_{\mathrm{class}}[\Phi] = \frac{1}{2} \int d\vec{x}\, (\nabla\Phi)^2 - \frac{M^2}{2\lambda'} A_1 + \frac{\lambda}{\lambda'^2} A_2. \tag{9}$$

It might be useful to note that among the divergent contributions (7) to the ground state energy there are no terms proportional to the kinetic part in the Lagrangian of the background $\Phi(x)$. Within the given framework this is a simple consequence of the structure of the heat kernel coefficients. Sometimes the first two terms of the perturbative expansion of $E_0[\Phi]$ with respect to powers of the background field Φ are considered as the divergent part of the ground state energy (e.g., in Ref. [14]). This has the disadvantage of containing nonlocal contributions (contrary to the heat kernel expansion which is local) and invites to discuss the kinetic term among the divergent ones which is in fact unnecessary.

As the renormalization procedure is not unique, a normalization condition must be imposed. We require

$$\lim_{m\to\infty} E_\varphi^{\mathrm{ren}}[\Phi] = 0. \tag{10}$$

For a massive field this is the natural condition requiring a vanishing energy of the quantum fluctuations when the field becomes heavy. As discussed in Ref. [15] this delivers a unique definition of the ground state energy (in contrast to the case of a massless field, see the discussion in Ref. [15] and also in Ref. [16]).

To proceed we follow Ref. [12] and use the representation of the regularized ground state energy in terms of the Jost function $f_l(k)$ of the scattering problem corresponding to Eq. (4),

$$E_\varphi[\Phi] = -\frac{\cos\pi s}{\pi} \mu^{2s} \sum_{l=0}^{\infty} (l+1/2)$$

$$\times \int_m^\infty dk [k^2 - m^2]^{1/2-s} \frac{\partial}{\partial k} \ln f_l(ik). \tag{11}$$

We remind the definition of the so-called regular solution $\phi_{n,l}(r)$ of the radial wave equation (see, e.g., Ref. [17]),

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r) + k^2 \right] \phi_{n,l}(r) = 0. \tag{12}$$

The regular solution behaves for $r\to 0$ as the free (i.e., without potential) solution. The Jost function (and its complex conjugate) are defined as the coefficients in the asymptotic expansion of this solution as $r\to\infty$:

$$\phi_{l,p}(r) \underset{r\to\infty}{\sim} \frac{i}{2} [f_l(p)\hat{h}_l^-(pr) - f_l^*(p)\hat{h}_l^+(pr)], \tag{13}$$

where $\hat{h}_l^-(pr)$ and $\hat{h}_l^+(pr)$ are the Riccati-Hankel functions.

Instead of dealing explicitly with eigenvalues, the vacuum energy is expressed in terms of the Jost function. Although representation (11) does not reveal explicitly the dependence of the energy on the bound states, it was shown in Ref. [12] that their contribution is correctly included. This fact, and furthermore the nonoscillating behavior of the Jost function on the imaginary axis [see Eq. (11)] gives the representation in our opinion calculational advantages with respect to alternative representations (e.g., integrating the scattering phases over the real k axis).

To proceed we have to perform the analytic continuation in s to $s=0$ in representation (8) of E_φ^{ren} using Eq. (11) for $E_\varphi[\Phi]$. This is not immediately possible because in the representation (11) the large l and k behavior of the summand leads to a divergent result. However, subtracting and adding the uniform asymptotic expansion $\ln f_k^{\mathrm{asym}}(ik)$ of $\ln f_l(ik)$ an analytical continuation can be obtained (for further details see Ref. [12]). As a result we get the sum of two parts, namely,

$$E_\varphi^{\mathrm{ren}}[\Phi] = E_f[\Phi] + E_{as}[\Phi] \tag{14}$$

with

$$E_{\mathrm{as}}[\Phi] = -\frac{\cos(\pi s)}{\pi}\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)$$

$$\times\int_m^{\infty}dk[k^2-m^2]^{\frac{1}{2}-s}\frac{\partial}{\partial k}\ln f_l^{\mathrm{asym}}(ik)-E_{\varphi}^{\mathrm{div}}.$$

(16)

$$E_f[\Phi] = -\frac{1}{\pi}\sum_{l=0}^{\infty}(l+1/2)\int_m^{\infty}dk[k^2-m^2]^{1/2}$$

$$\times\frac{\partial}{\partial k}[\ln f_l(ik)-\ln f_l^{\mathrm{asym}}(ik)]$$

(15)

and

The first contribution $E_f[\Phi]$ is called the *finite* part. The regularization parameter s has already been put to zero because the sum and the integral are convergent due to the subtraction. The second contribution $E_{\mathrm{as}}[\Phi]$ is called the *asymptotic* contribution. By construction it has a finite limit for $s\to 0$ which is determined below. We will need the asymptotic expansion $\ln f_l^{\mathrm{asym}}(ik)$ for $\nu\to\infty$, $k\to\infty$, ν/k fixed. Up to the required order of ν^{-3} it reads [12]

$$\ln f_l^{\mathrm{asym}}(ik)=\frac{1}{2\nu}\int_0^{\infty}dr\frac{r\,V(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{1/2}}+\frac{1}{16\nu^3}\int_0^{\infty}dr\frac{rV(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{3/2}}\left[1-\frac{6}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]}+\frac{5}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^2}\right]$$

$$-\frac{1}{8\nu^3}\int_0^{\infty}dr\frac{r^3\,V^2(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{3/2}},$$

(17)

with $\nu=l+1/2$.

III. THE ASYMPTOTIC CONTRIBUTION

First of all the k-integrals in Eq. (16) can be done explicitly using

$$\int_m^{\infty}dk[k^2-m^2]^{1/2-s}\frac{\partial}{\partial k}\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{-n/2}=-\frac{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\Gamma(n/2)}\frac{\left(\frac{\nu}{mr}\right)^n m^{1-2s}}{\left(1+\left(\frac{\nu}{mr}\right)^2\right)^{s+(n-1)/2}}.$$

(18)

This naturally leads to functions of the type

$$f(s;a;b)=\sum_{\nu=1/2,3/2,\ldots}^{\infty}\nu^a\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s-b},$$

(19)

which need to be evaluated around $s=-1/2$. The way to deal with these functions has been developed in Ref. [18] and for completeness the relevant results are provided in the Appendix, Eq. (A3). Using these, it is a simple exercise to find

$$E_{\mathrm{as}}^{\mathrm{ren}}[\Phi]=-\frac{1}{8\pi}\int_0^{\infty}dr\,r^2V^2(r)\ln(mr)-\frac{1}{2\pi}\int_0^{\infty}dr\,V(r)\int_0^{\infty}d\nu\frac{\nu}{1+e^{2\pi\nu}}\ln|\nu^2-(mr)^2|$$

$$-\frac{1}{8\pi}\int_0^{\infty}dr\left[r^2V^2(r)-\frac{1}{2}V(r)\right]\int_0^{\infty}d\nu\left(\frac{d}{d\nu}\frac{1}{1+e^{2\pi\nu}}\right)\ln|\nu^2-(mr)^2|$$

$$-\frac{1}{8\pi}\int_0^{\infty}dr\,V(r)\int_0^{\infty}d\nu\left[\frac{d}{d\nu}\left(\frac{1}{\nu}\frac{d}{d\nu}\frac{\nu^2}{1+e^{2\pi\nu}}\right)\right]\ln|\nu^2-(mr)^2|$$

$$+\frac{1}{48\pi}\int_0^{\infty}dr\,V(r)\int_0^{\infty}d\nu\left[\frac{d}{d\nu}\left(\frac{1}{\nu}\frac{d}{d\nu}\frac{1}{\nu}\frac{d}{d\nu}\frac{\nu^4}{1+e^{2\pi\nu}}\right)\right]\ln|\nu^2-(mr)^2|,$$

(20)

suitable for numerical evaluation.

IV. NUMERICAL DETERMINATION OF THE JOST FUNCTION AND THE INTEGRATION PROCEDURE

Under the assumption that the potential has compact support, let's say $V(r)=0$ for $r \geq R$, the regular solution may be written as

$$\phi_{l,p}(r)=u_{l,p}(r)\Theta(R-r)+\frac{i}{2}[f_l(p)\hat{h}_l^-(pr)$$

$$-f_l^*(p)\hat{h}_l^+(pr)]\Theta(r-R). \qquad (21)$$

This solution and its derivative must be continuous at $r=R$ leading to the matching conditions

$$u_{l,p}(R)=\frac{i}{2}[f_l(p)\hat{h}_l^-(pR)-f_l^*(p)\hat{h}_l^+(pR)],$$

$$u_{l,p}'(R)=\frac{i}{2}p[f_l(p)\hat{h}_l^{-\prime}(pR)-f_l^*(p)\hat{h}_l^{+\prime}(pR)].$$

From these conditions we get the Jost function in terms of the regular solutions

$$f_l(p)=-\frac{1}{p}[pu_{l,p}(R)\hat{h}_l^{+\prime}(pR)-u_{l,p}'(R)\hat{h}_l^+(pR)],$$

$$(22)$$

where we used that the Wronskian determinant of $\hat{h}_l^\pm$ is $2i$.

In order to determine a unique solution of the differential equation (12), we need to pose the initial value problem corresponding to the regular solution $\phi_{l,p}(r)$. We have, for $r \to 0$,

$$\phi_{l,p}(r)=u_{l,p}(r) \sim \hat{j}_l(pr) \sim \frac{\sqrt{\pi}}{\Gamma(l+3/2)}\left(\frac{pr}{2}\right)^{l+1}.$$

We change to the new function

$$u_{l,p}(r)=\frac{\sqrt{\pi}}{\Gamma(l+3/2)}\left(\frac{pr}{2}\right)^{l+1}g_{l,p}(r)$$

with the initial value $g_{l,p}(0)=1$. The differential equation for $g_{l,p}(r)$ reads

$$\left\{\frac{d^2}{dr^2}+2\frac{l+1}{r}\frac{d}{dr}-V(r)+p^2\right\}g_{l,p}(r)=0$$

or, on the imaginary p-axis written as a first order differential equation with $(\partial/\partial r)g_{l,ip}(r)=v_{l,ip}(r)$,

$$\frac{d}{dr}\binom{g_{l,ip}(r)}{v_{l,ip}(r)}=\begin{pmatrix} 0 & 1 \\ V(r)+p^2 & -\frac{2}{r}(l+1) \end{pmatrix}\binom{g_{l,ip}(r)}{v_{l,ip}(r)}.$$

$$(23)$$

To fix the solution uniquely we only need to fix $v_{l,ip}(0)$. A power series ansatz for $g_{l,ip}(r)$ about $r=0$ shows that for $V(r)=\mathcal{O}(r^{-1+\epsilon})$ the condition reads $v_{l,ip}(0)=0$. With this unique solution of Eq. (23), the Jost function takes the form

$$f_l(ip)=\frac{2}{\Gamma(l+3/2)}\left(\frac{pR}{2}\right)^{l+3/2}\Big\{g_{l,ip}(R)K_{l+3/2}(pR)$$

$$+\frac{1}{p}g_{l,ip}'(R)K_{l+1/2}(pR)\Big\}. \qquad (24)$$

Finally, after a partial integration and the substitution $q=\sqrt{k^2-m^2}$ in Eq. (16), the starting point for the numerical evaluation used for E_f is

$$E_f=\frac{1}{\pi}\sum_{l=0}^{\infty}(l+1/2)\int_0^{\infty}dq[\ln f_l(i\sqrt{q^2+m^2})$$

$$-\ln f_l^{\mathrm{asym}}(i\sqrt{q^2+m^2})], \qquad (25)$$

where $\ln f_l^{\mathrm{asym}}$ is given by Eq. (17).

V. EXAMPLES

The method described above works for radially symmetric potentials $\Phi(r)$ having compact support $\Phi(r \geq R)=0$. In order to have a finite classical energy, Eq. (9), we also demand $\Phi'(r=0)=\Phi'(r=R)=0$.

In the following we use dimensionless quantities:

$$\epsilon=ER, \qquad (26)$$

$$\mu=mR,$$

$$\rho=\frac{r}{R},$$

$$V(r)=\lambda'\Phi^2(r)=\frac{1}{R^2}v(\rho)=\frac{\lambda'}{R^2}\varphi^2(\rho)$$

such that $\epsilon=\epsilon[\varphi,\mu]$. Therefore, large μ corresponds to large R or to large m.

We studied two types of potentials. Type A is lumplike concentrated around $r=0$ whereas type B has its maximum at $R/2$, acting as a spherical wall:

$$\varphi_A(\rho)=\frac{b(1-\rho^2)^2}{b+\rho^2}, \qquad (27)$$

$$\varphi_B(\rho)=\frac{16b\rho^2(1-\rho)^2}{b+(1-2\rho)^2}.$$

The parameter b allows to vary the shape of the potentials, as indicated in the insets of the figures.

Our numerical implementation of Eq. (25) showed that the sum over angular momenta converges quite fast, usually after 10–50 terms. Hereby, the numerical q-integration pro-
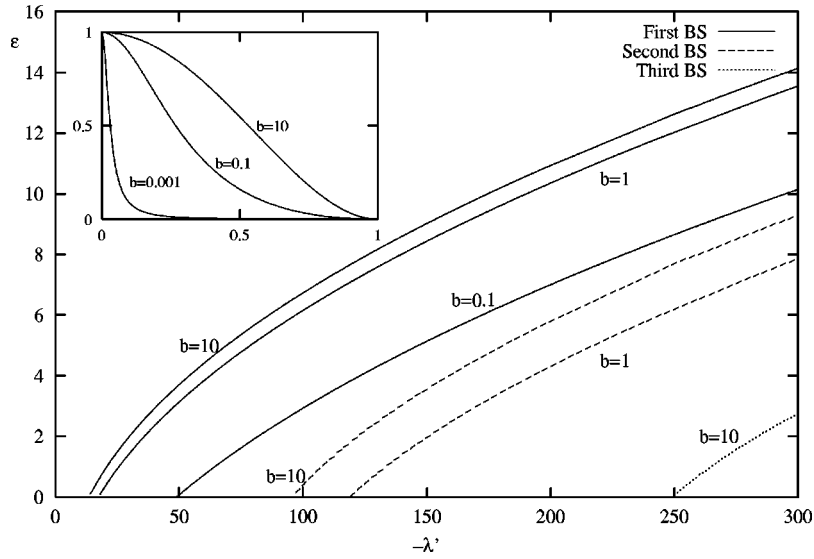
FIG. 1. Binding energy of bound states (BS) for the type A potential and negative λ' [the inset shows $\varphi(\rho)$ where $V(\rho)=\lambda'\varphi^2(\rho)$].

cedure has to solve the system of ordinary differential equations Eq. (23) for every evaluation of the Jost function $f_l(ip)$.

As a check, we compared our results with the asymptotic formulas

$$\epsilon \sim \begin{cases} \dfrac{1}{16\pi^2}A_2\log\dfrac{1}{\mu} & \text{for } \mu\to 0, \\[2ex] \dfrac{-1}{32\pi^2}\dfrac{A_3}{\mu^2} & \text{for } \mu\to\infty, \end{cases} \qquad (28)$$

where $A_2=\frac{1}{2}\int d\vec{x}\, V(r)^2$ is the second heat kernel coefficient and $A_3=\frac{1}{6}\int d\vec{x}[\frac{1}{2}V(r)\Delta V(r)-V(r)^3]$ is the third one. These relations follow from scaling arguments together with the well known property of the heat kernel expansion being the asymptotic expansion for large masses.

The most determining property of the potential $V(r)$ in the attractive case $\lambda'<0$ is the number and depth of bound states. The parameter dependence of the $l=0$ bound states, determined as zeroes of the $l=0$ Jost function, Eq. (24), is shown in Figs. 1 and 2. Number as well as depth of the bound states increases with increasing $|\lambda'|$ and with increasing b, too.

As Eq. (25) shows, the Casimir energy develops an imaginary part (corresponding to particle creation by the external field) when the q integration passes a zero of the Jost function. E_f is real as long as μ is larger than the depth of the highest bound state. Therefore we indicate in our Figs. 3–6 the positions of bound states on the μ axis by vertical lines where the plots of the Casimir energy end.

To show clearly the influence of the form and spectrum of V on the vacuum energy, we consider the dependence of the energy for fixed λ' and different values of the parameter b,
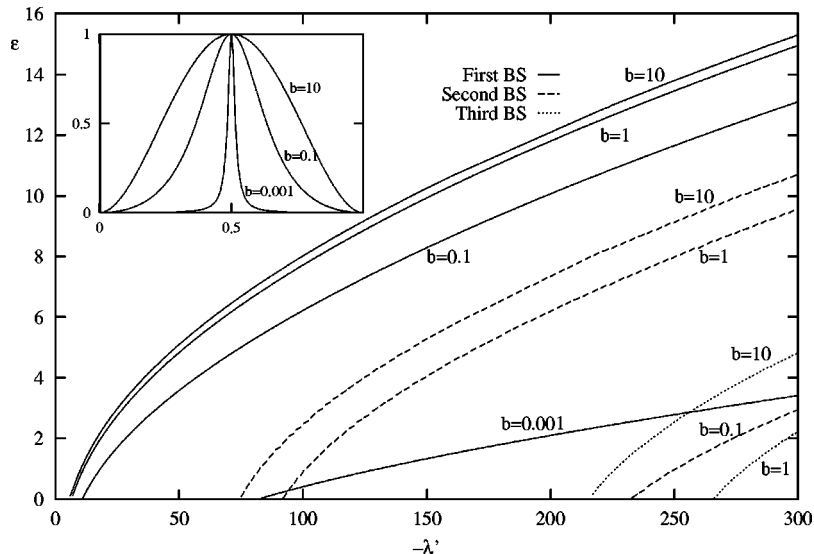

FIG. 2. Binding energy of bound states for the type B potential and negative λ' [the inset shows $\varphi(\rho)$ where $V(\rho)=\lambda'\varphi^2(\rho)$].
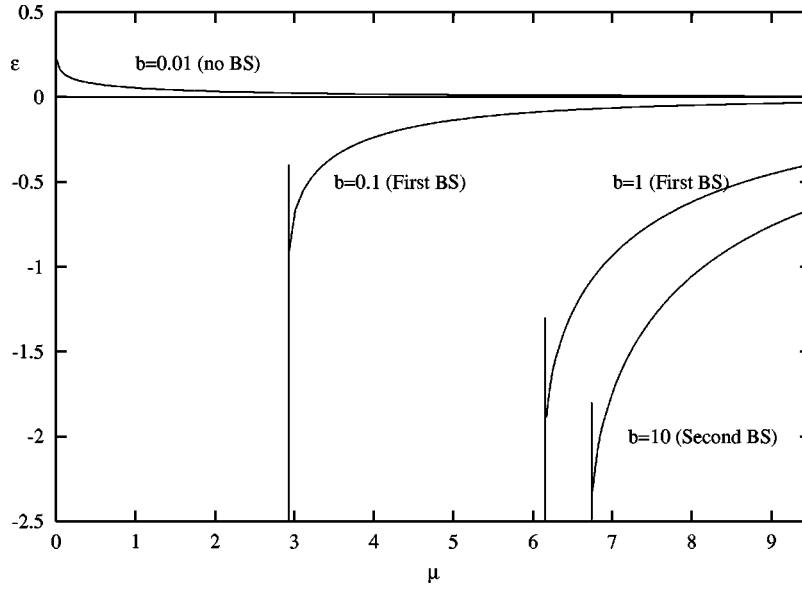
FIG. 3. Casimir energy for type A potentials of different shapes with $\lambda' = -100$. The positions of bound states at the μ axis are shown as vertical lines.

Figs. 3 and 4, as well as for fixed shape b and different depths λ' in Figs. 5 and 6. These figures demonstrate that the vacuum energy is determined by two competing contributions, the positive contribution of the scattering states and the negative contribution of the bound states of the potential.

Let us start with a description and interpretation of Figs. 3 and 4. For b large, the contribution of the bound state(s) to the vacuum energy is large enough to overcompensate the contribution of the scattering states and the energy is negative. At some "critical value" of b, the energy of the only remaining bound state is so small that the positive contributions of the scattering states get the upper hand, and the vacuum energy is positive. Finally, for $b \to 0$ we have by normalization $E^{\text{ren}}_{\varphi}[\Phi = 0] = 0$, such that at some point E^{ren}_{φ} starts to decrease again with decreasing b.

The same features are clearly recovered in Figs. 5 and 6. For $(-\lambda')$ large enough a negative vacuum energy is obtained. With decreasing $(-\lambda')$ the energy increases and can be positive or negative depending on the parameter μ. Again, at some point the scattering states dominate, the energy gets positive and tends to zero as λ' tends to zero. For positive λ' the energy starts to increase again, being virtually identical for the two values $\lambda' = \pm 2$, and it seems that E^{ren}_{φ} is a monotonically increasing function of λ' for $\lambda' \geq 0$. Since we see no qualitative difference between our two potential types A and B, we expect these features to be universal. Given that the dependence of the energy on λ' is somewhat nontrivial, let us show this dependence explicitly for type A potentials in Fig. 7.
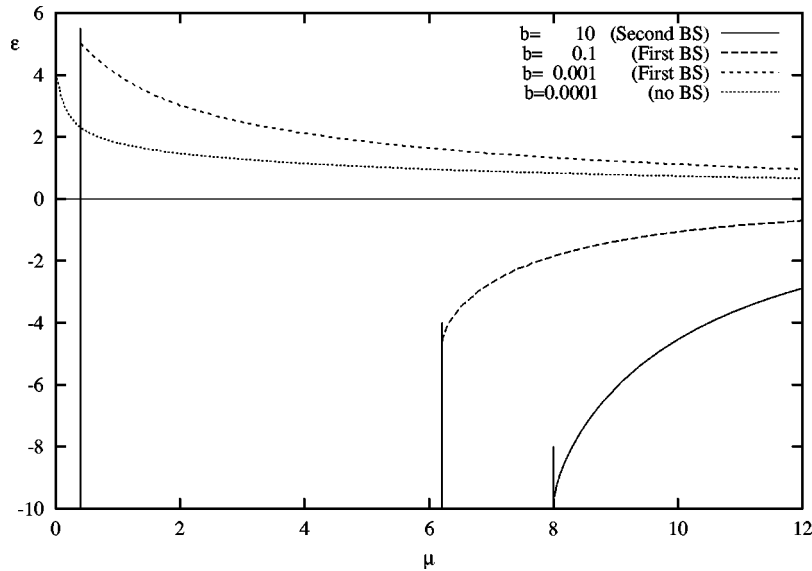

FIG. 4. Casimir energy for type B potentials of different shapes with $\lambda' = -100$. The positions of bound states (BS) at the μ axis are shown as vertical lines.
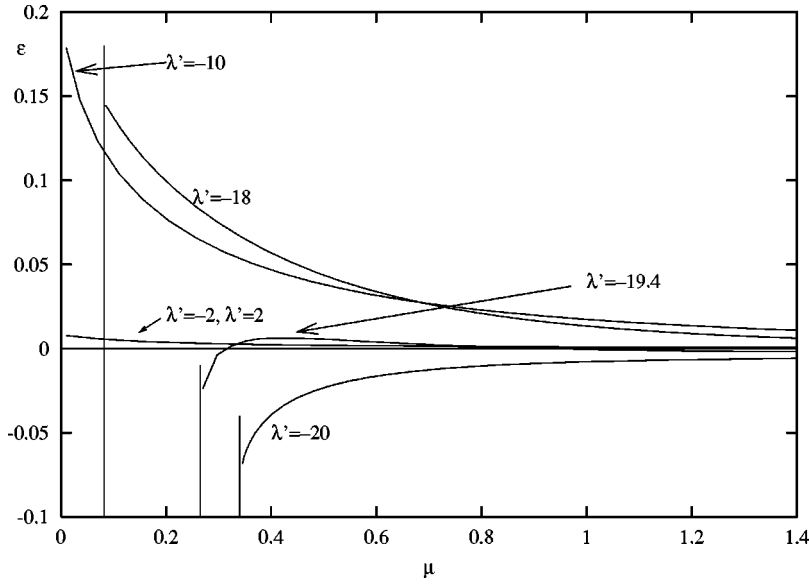
FIG. 5. Casimir energy for type A potentials of different magnitudes λ' with equal shape parameter $b=1$. The positions of bound states existing for $\lambda'=-18,-19.4$, and -20 at the μ axis are shown as vertical lines.

As long as no bound states exist, the sign of λ' is not important and the energy is nearly symmetric about $\lambda'=0$. The contribution to the energy of the scattering states does not depend much on the sign of λ'. The symmetry is disturbed as soon as a bound state shows up and the energy then decreases with decreasing λ' because the bound state depth as well as the number of bound states increases.

VI. CONCLUSIONS

We demonstrated that our method is quite feasible for calculations of the Casimir energy in realistic three-dimensional examples for arbitrary spherically symmetric external potentials as long as they are smooth and have compact support.

The behavior of the Casimir energy shows some interesting features. The existence of bound states is a necessary but by no means sufficient condition for a negative vacuum energy. Even in the presence of bound states the sign of the energy can depend on the scale parameter μ, i.e., the mass of the quantum field measured in units of the size R of the potential.

For shallow potentials without bound states (small $|\lambda'|$) the Casimir energy is nearly invariant under a change of the sign of the potential whereas for deep attractive potentials the negative contribution of the bound states dominates the vacuum energy.
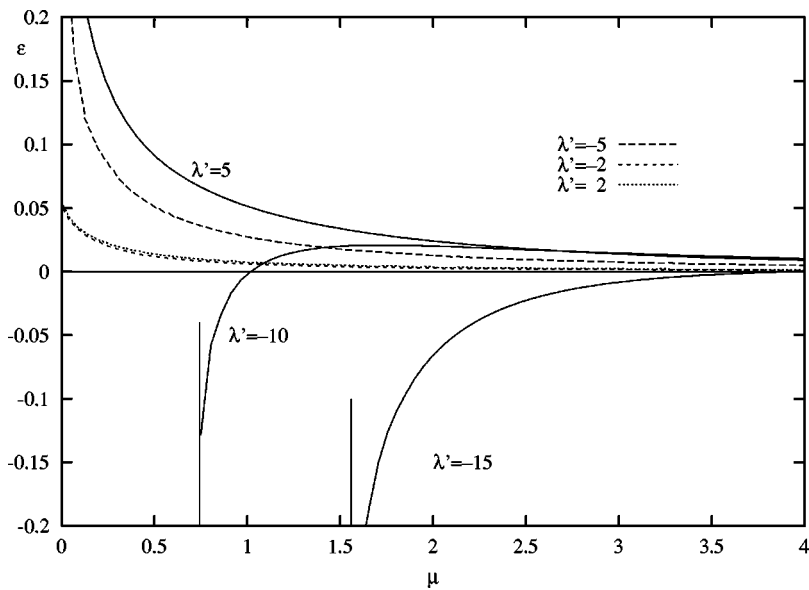

FIG. 6. Casimir energy for type B potentials of different magnitudes λ' with equal shape parameter $b=1$. The positions of bound states existing for $\lambda'=-15$ and -10 at the μ axis are shown as vertical lines.
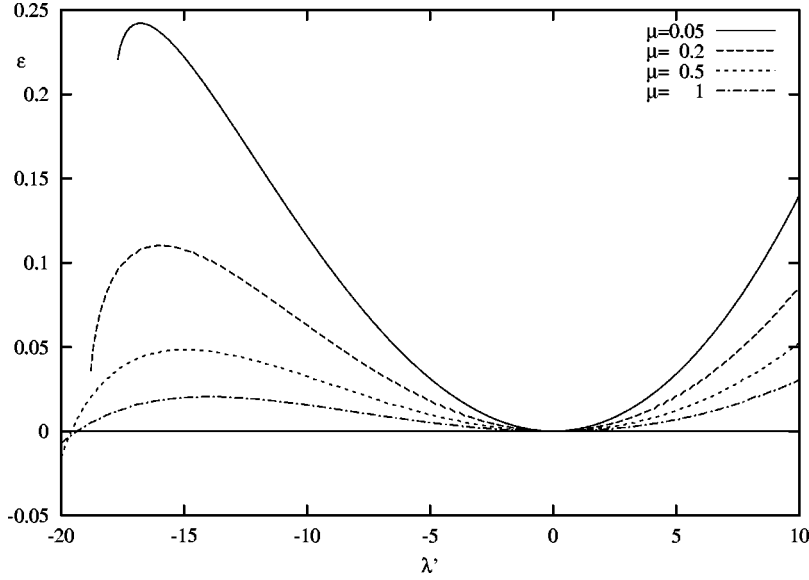
FIG. 7. Casimir energy for type A potentials for different magnitudes λ' and equal shape parameter $b=1$ as a function of the parameter μ.

On the other hand we found no qualitative differences between a lumplike and a ringlike classical background configuration. Further, the potentials considered have a power-like decrease at spatial infinity. However, in the numerical procedure they are cut at some finite radius so that they become effectively of finite size. We found that the numerical results are insensitive to this. The method developed here can be used for the calculation of quantum corrections to classical objects such as Skyrmions and others.

APPENDIX: ANALYTICAL CONTINUATION OF THE ASYMPTOTIC CONTRIBUTIONS

The essential formulas for the basic series $f(s;a;b)$, Eq. (19), in the calculation are the following (we use $x=mr$):

$$\sum_{\nu=1/2,3/2,\ldots}^{\infty} \nu^{2n+1}\left(1+\left(\frac{\nu}{x}\right)^2\right)^{-s} = \frac{1}{2}\frac{n!\,\Gamma(s-n-1)}{\Gamma(s)}x^{2n+2}+(-1)^n 2\int_0^x d\nu\frac{\nu^{2n+1}}{1+e^{2\pi\nu}}\left[1-\left(\frac{\nu}{x}\right)^2\right]^{-s}$$

$$+(-1)^n 2\cos(\pi s)\int_x^{\infty}d\nu\frac{\nu^{2n+1}}{1+e^{2\pi\nu}}\left[\left(\frac{\nu}{x}\right)^2-1\right]^{-s}, \tag{A1}$$

$$\sum_{\nu=1/2,3/2,\ldots}^{\infty} \nu^{2n}\left[1+\left(\frac{\nu}{x}\right)^2\right]^{-s} = \frac{1}{2}\frac{\Gamma(n+1/2)\Gamma(s-n-1/2)}{\Gamma(s)}x^{2n+1}-(-1)^n 2\sin(\pi s)\int_x^{\infty}d\nu\frac{\nu^{2n}}{1+e^{2\pi\nu}}\left[\left(\frac{\nu}{x}\right)^2-1\right]^{-s}. \tag{A2}$$

Using partial integrations one can get representations valid for all values of s needed. The explicit results applied for the calculation of Eq. (16) are [we shall use the notation $f(a;b)=f(-1/2;a;b)$],

$$f(1;1/2)=-\frac{1}{2}x^2+\frac{1}{24},$$

$$\frac{d}{ds}\Bigg|_{s=-1/2}f(s;1;1/2)=-\frac{1}{2}x^2-2\int_0^{\infty}d\nu\frac{\nu}{1+e^{2\pi\nu}}\ln\left|1-\left(\frac{\nu}{x}\right)^2\right|,$$

$$f(1;3/2) = \frac{x^2}{2(s+1/2)} + x^2 \ln x + x^2 \int_0^\infty d\nu \left(\frac{d}{d\nu} \frac{1}{1+e^{2\pi\nu}} \right) \ln|\nu^2 - x^2|,$$

$$f(3;5/2) = \frac{x^4}{2(s+1/2)} + (\ln x - 1/2)x^4 + \frac{x^4}{2} \int_0^\infty d\nu \left[\frac{d}{d\nu} \left(\frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^2}{1+e^{2\pi\nu}} \right) \right] \ln|\nu^2 - x^2|,$$

$$f(5;7/2) = \frac{x^6}{2(s+1/2)} + (\ln x - 3/4)x^6 + \frac{x^6}{8} \int_0^\infty d\nu \left[\frac{d}{d\nu} \left(\frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^4}{1+e^{2\pi\nu}} \right) \right] \ln|\nu^2 - x^2|, \tag{A3}$$

which provide the needed expansions about $s=0$ in (16).

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[1] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[2] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).

[3] A. M. Polyakov, JETP Lett. **20**, 194 (1974).

[4] F. R. Klinkhamer and N. S. Manton, Phys. Rev. D **30**, 2212 (1984).

[5] T. H. R. Skyrme, Nucl. Phys. **B31**, 556 (1962).

[6] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).

[7] A. A. Actor and I. Bender, Phys. Rev. D **52**, 3581 (1995).

[8] M. Bordag, D. Hennig, and D. Robaschik, J. Phys. A **25**, 4483 (1992).

[9] Joon-Il Kim and Byung-Yoon Park, Phys. Rev. D **57**, 2853 (1998).

[10] Norberto N. Scoccola and Hans Walliser, Phys. Rev. D **58**, 094037 (1998).

[11] M. Bordag, J. Phys. A **28**, 755 (1995).

[12] M. Bordag and K. Kirsten, Phys. Rev. D **53**, 5753 (1996).

[13] J. Baacke, Z. Phys. C **47**, 263 (1990).

[14] E. Farhi, N. Graham, P. Haagensen, and R. L. Jaffe, Phys. Lett. B **427**, 334 (1998).

[15] M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999).

[16] Steve Blau, Matt Visser, and Andreas Wipf, Nucl. Phys. **B310**, 163 (1988).

[17] J.R. Taylor, *Scattering Theory* (Wiley, New York, 1972).

[18] E. Elizalde, M. Bordag, and K. Kirsten, J. Phys. A **31**, 1743 (1998).